PACKING 3-VERTEX PATHS IN CUBIC 3-CONNECTED GRAPHS

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Abstract

Let $v(G)$ and $\lambda(G)$ be the number of vertices and the maximum number of disjoint 3-vertex paths in G, respectively. We discuss the following old

Problem. *Is the following claim true ?*
(P) *if* G *is a 3-connected and cubic graph, then* $\lambda(G) = \lfloor v(G)/3 \rfloor$.

We show, in particular, that claim (P) is equivalent to some seemingly stronger claims (see **3.1**). It follows that if claim (P) is true, then Reed's dominating graph conjecture is true for cubic 3-connected graphs.

Keywords: cubic 3-connected graph, 3-vertex path packing, 3-vertex path factor, domination.

1 Introduction

We consider undirected graphs with no loops and no parallel edges. All notions and facts on graphs, that are used but not described here, can be found in [1, 2, 15].

Given graphs G and H, an *H-packing* of G is a subgraph of G whose components are isomorphic to H. An *H-packing* P of G is called an *H-factor* if $V(P) = V(G)$. The *H-packing problem*, i.e. the problem of finding in G an H-packing, having the maximum number of vertices, turns out to be NP-hard if H is a connected graph with at least three vertices [3]. Let Λ denote a 3-vertex path. In particular, the Λ-packing problem is NP-hard. Moreover, this problem remains NP-hard even for cubic graphs [5].

Although the Λ-packing problem is NP-hard, i.e. possibly intractable in general, this problem turns out to be tractable for some natural classes of graphs. It would be also interesting to find polynomial algorithms that would provide a good approximation solution for the problem. Below (see **1.3**, **1.1**, and **1.2**) are some examples of such results. In each case the corresponding packing problem is polynomially solvable.

Let $v(G)$ and $\lambda(G)$ denote the number of vertices and the maximum number of disjoint 3-vertex paths in G, respectively. Obviously $\lambda(G) \le \lfloor v(G)/3 \rfloor$.

A graph is called *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$ (which is called a *claw*). A block of a connected graph is called an *end-block* if it has at most one vertex in common with any other block of the graph. Let $eb(G)$ denote the number of end-blocks of G.

1.1 [11] *Suppose that G is a connected claw-free graph and $eb(G) \ge 2$. Then $\lambda(G) \ge \lfloor (v(G) - eb(G) + 2)/3 \rfloor$, and this lower bound is sharp.*

1.2 [11] *Suppose that G is a connected and claw-free graph having at most two end-blocks (in particular, a 2-connected and claw-free graph). Then $\lambda(G) = \lfloor v(G)/3 \rfloor$.*

Obviously the claim in **1.2** on claw-free graphs with exactly two end-blocks follows from **1.1**.

In [4, 12] we answered the following natural question:

How many disjoint 3-vertex paths must a cubic graph have?

1.3 *If G is a cubic graph then $\lambda(G) \ge \lceil v(G)/4 \rceil$. Moreover, there is a polynomial time algorithm for finding a Λ-packing having at least $\lceil v(G)/4 \rceil$ components.*

Obviously if every component of G is K_4, then $\lambda(G) = v(G)/4$. Therefore the bound in **1.3** is sharp.

Let $\mathcal{G}_2^3$ denote the set of graphs with each vertex of degree at least 2 and at most 3. In [4] we answered (among other results) the following question:

How many disjoint 3-vertex paths must an n-vertex graph from $\mathcal{G}_2^3$ have?

1.4 *Suppose that $G \in \mathcal{G}_2^3$ and G has no 5-vertex components. Then $\lambda(G) \ge \lceil v(G)/4 \rceil$.*

Obviously **1.3** follows from **1.4** because if G is a cubic graph, then $G \in \mathcal{G}_2^3$ and G has no 5-vertex components.

In [4] we also gave a construction that allowed to prove the following:

1.5 *There are infinitely many connected graphs for which the bound in* **1.4** *is attained. Moreover, there are infinitely many subdivisions of cubic 3-connected graphs for which the bound in* **1.4** *is attained.*

The next interesting question is:

How many disjoint 3-vertex paths must a cubic connected graph have?

In [6] we proved the following.

1.6 *Let $\mathcal{C}(n)$ denote the set of connected cubic graphs with n vertices and $\lambda_n = \min\{\lambda(G)/v(G) : G \in \mathcal{C}(n)\}$. Then for some $c > 0$,*

$$\frac{3}{11}(1 - \frac{c}{n}) \le \lambda_n \le \frac{3}{11}(1 - \frac{1}{n^2}).$$

The next natural question is:

1.7 Problem *How many disjoint 3-vertex paths must a cubic 2-connected graph have?*

This question is still open (namely, the sharp lower bound on the number of disjoint 3-vertex paths in a cubic 2-connected n-vertex graph is unknown).

On the other hand, it is also natural to consider the following

1.8 Problem. *Are there 2-connected cubic graphs G such that $\lambda(G) < \lfloor v(G)/3 \rfloor$?*

In [7] we gave a construction that provided infinitely many 2-connected, cubic, bipartite, and planar graphs such that $\lambda(G) < \lfloor v(G)/3 \rfloor$.

The main goal of this paper (see also [8]) is to discuss the following old open problem which is similar to Problem **1.8**.

1.9 Problem. *Is the following claim true ?*

(P) *if G is a 3-connected and cubic graph, then $\lambda(G) = \lfloor v(G)/3 \rfloor$.*

We show, in particular, that claim (P) in **1.9** is equivalent to some seemingly stronger claims (see **3.1**).

In Section 2 we give some notation, constructions, and simple observations.

In Section 3 we formulate and prove our main theorem **3.1** concerning various claims that are equivalent to claim (P) in **1.9**. We actually give different proofs of **3.1**. Thus if there is a counterexample C to one of the above claims, then the different proofs below provide different constructions of counterexamples to the other claims in **3.1**. Moreover, different proofs provide better understanding of relations between various Λ-packing properties considered in **3.1**.

In [14] B. Reed conjectured that if G is a connected cubic graph, then $\gamma(G) \le \lceil v(G)/3 \rceil$, where $\gamma(G)$ is the dominating number of G (i.e. the size of a minimum vertex subset X in G such that every vertex in $G - X$ is adjacent to a vertex in X). It turns out that Reed's conjecture is not true for connected and even for 2-connected cubic graphs [9, 13]. If claim (P) in **1.9** is true, then from **3.1** it follows, in particular, that Reed's conjecture is true for 3-connected cubic graphs.

In Section 4 we describe some results showing that certain claims in **3.1** are best possible.

In Section 5 we give a Λ-factor homomorphism theorem in cubic graphs.

2 Notation, constructions, and simple observations

We consider undirected graphs with no loops and no parallel edges unless stated otherwise. As usual, $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively, and $v(G) = |V(G)$. If X is a vertex subset or a subgraph of G, then let $D(X, G)$ or simply $D(X)$, denotes the set of edges in G, having exactly one end–vertex in X, and let $d(X, G) = |D(X, G)|$. If $x \in V(G)$, then $D(x, G)$ is the set of edges in G incident to x, $d(x, G) = |D(x, G)|$, $N(x, G) = N(x)$ is the set of vertices in G adjacent to x, and

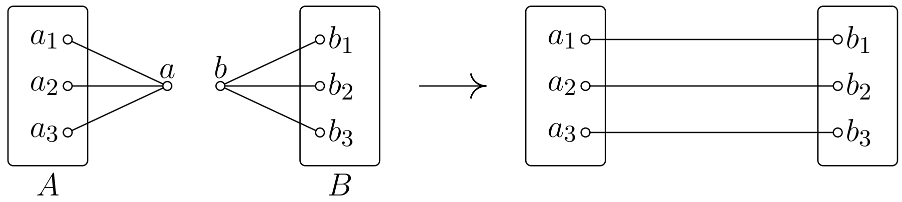

Figure 1: $Aa\sigma bB$

$\Delta(G) = \max\{d(x, G) : x \in V(G)\}$. If $e = xy \in E(G)$, then let $End(e) = \{x, y\}$. Let $Cmp(G)$ denote the set of components of G and $cmp(G) = |Cmp(G)|$.

Let A and B be disjoint graphs, $a \in V(A)$, $b \in V(B)$, and $\sigma : N(a, A) \to N(b, B)$ be a bijection. Let $Aa\sigma bB$ denote the graph $(A - a) \cup (B - b) \cup \{x\sigma(x) : x \in N(a, A)\}$. We usually assume that $N(a, A) = \{a_1, a_2, a_3\}$, $N(b, B) = \{b_1, b_2, b_3\}$, and $\sigma(a_i) = b_i$ for $i \in \{1, 2, 3\}$ (see Fig. 1). We also say that *$Aa\sigma bB$ is obtained from B by replacing vertex b by $(A - a)$ according to σ.*

Let B be a cubic graph and $X \subseteq V(B)$. Let $A(v)$, where $v \in X$, be a graph, a^v be a vertex of degree three in $A(v)$, and $A^v = A(v) - a^v$. By using the above operation, we can build a graph $G = B\{(A(v), a^v) : v \in X\}$ by replacing each vertex v of B in X by A^v assuming that all $A(v)$'s are disjoint. Let $D^v = D(A^v, G)$. For each $u \in V(B) \setminus X$ let $A(u)$ be the graph having exactly two vertices u, a^u and exactly three parallel edges connecting u and a^u. Then $G = B\{(A(v), a^v) : v \in X\} = B\{(A(v), a^v) : v \in V(B)\}$. If, in particular, $X = V(G)$ and each $A(v)$ is a copy of K_4, then G is obtained from B by replacing each vertex by a triangle.

Let $E' = E(G) \setminus \cup\{E(A^v) : v \in V(B)\}$. Obviously, there is a unique bijection $\alpha : E(B) \to E'$ such that if $uv \in E(B)$, then $\alpha(uv)$ is an edge in G having one end-vertex in A^u and the other in A^v.

Let P be a Λ-packing in G. For $uv \in E(B)$, $u \neq v$, we write $u \neg^p v$ or simply, $u \neg v$, if P has a 3-vertex path L such that $\alpha(uv) \in E(L)$ and $|V(A^u) \cap V(L)| = 1$. Let P^v be the union of components of P that meet D^v in G.

Obviously

2.1 *Let k be an integer and $k \leq 3$. If A and B above are k-connected, cubic, bipartite, and planar graphs, then $Aa\sigma bB$ is also a k-connected, cubic, bipartite, and planar graph, respectively.*

From **2.1** we have:

2.2 *Let k be an integer and $k \leq 3$. If B and each A_v is a k-connected, cubic, bipartite, and planar graphs, then $B\{(A_v, a_v) : v \in V(B)\}$ is also a k-connected, cubic, bipartite, and planar graph, respectively.*

Let A^1, A^2, A^3 be three disjoint graphs, $a^i \in V(A^i)$, and $N(a^i, A^i) = \{a^i_1, a^i_2, a^i_3\}$, where $i \in \{1, 2, 3\}$. Let $F = Y(A^1, a^1; A^2, a^2; A^3, a^3)$ denote the graph obtained from

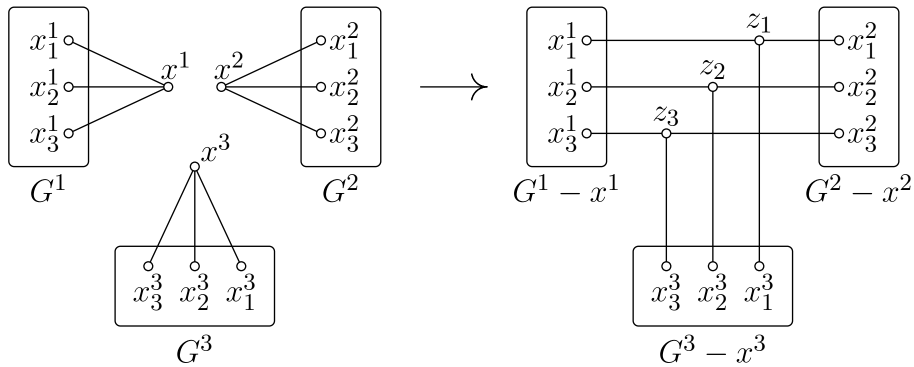

Figure 2: $Y(A^1, a^1; A^2, a^2; A^3, a^3)$

$(A^1 - a^1) \cup (A^2 - a^2) \cup (A^3 - a^3)$ by adding three new vertices z_1, z_2, z_3 and the set of nine new edges $\{z_j a^i_j : i, j \in \{1, 2, 3\}$ (see Fig. 2). In other words, if $B = K_{3,3}$ is the complete (X, Z)-bipartite graph with $X = \{x_1, x_2, x_3\}$ and $Z = \{z_1, z_2, z_3\}$, then F is obtained from the B by replacing each vertex x_i in X by $A^i - a^i$ so that $D(A^i - a^i, F) = \{a^i_j z_j : j \in \{1, 2, 3\}$. Let $D^i = D(A^i - a^i, F)$. If P is a Λ-packing of F, then let $P^i = P^i(F)$ be the union of components of P meeting D^i and $E^i = A^i(P) = E(P) \cap D^i$, $i \in \{1, 2, 3\}$.

If each (A^i, a^i) is a copy of the same (A, a), then we write $Y(A, a)$ instead of $Y(A^1, a^1; A^2, a^2; A^3, a^3)$.

From **2.2** we have, in particular:

2.3 *Let k be an integer and $k \le 3$. If each A^i above is a k-connected, cubic, and bipartite graph, then* $Y(A^1, a^1; A^2, a^2; A^3, a^3)$ (see Fig. 2) *is also a k-connected, cubic, and bipartite graph, respectively.*

We will use the following simple observation.

2.4 *Let A and B be disjoint graphs, $a \in V(A)$, $N(a, A) = \{a_1, a_2, a_3\}$, $b \in V(B)$, $N(b, B) = \{b_1, b_2, b_3\}$, and $G = Aa\sigma bB$, where each* $\sigma(a_i) = b_i$ (see Fig. 1). *Let P be a Λ-factor of G (and so $v(G) = 0 \bmod 3$) and P' be the Λ-packing of G consisting of the components (3-vertex paths) of P that meet $\{a_1 b_1, a_2 b_2, a_3 b_3\}$.*

(a1) Suppose that $v(A) = 0 \bmod 3$, and so $v(B) = 2 \bmod 3$. Then one of the following holds (see Fig 3)*:*

(a1.1) P' has exactly one component that has two vertices in $A - a$, that are adjacent (and, accordingly, exactly one vertex in $B - b$),

(a1.2) P' has exactly two components and each component has exactly one vertex in $A - a$ (and, accordingly, exactly two vertices in $B - b$, that are adjacent),

(a1.3) P' has exactly three components L_1, L_2, L_3 and one of them, say L_1, has exactly one vertex in $A - a$ and each of the other two L_2, L_3, has exactly two vertices in $A - a$, that are adjacent (and, accordingly, L_1 has exactly two vertices in $B - b$, that

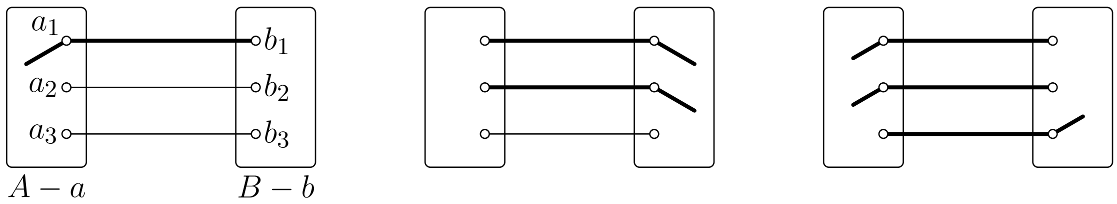

Figure 3:

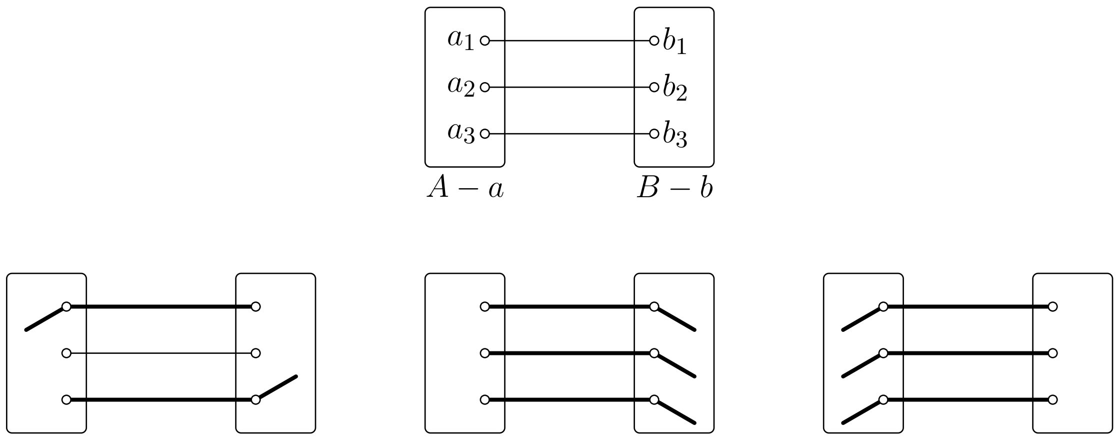

Figure 4:

are adjacent, and each of the other two L_2, L_3, has exactly one vertex in $B-b$).

($a2$) *Suppose that $v(A) = 1 \bmod 3$, and so $v(B) = 1 \bmod 3$. Then one of the following holds* (see Fig 4)*:*

($a2.1$) $P' = \emptyset$,

($a2.2$) *P' has exactly two components, say L_1, L_2, and one of the them, say L_1, has exactly one vertex in $A-a$ and exactly two vertices in $B-b$, that are adjacent, and the other component L_2 has exactly two vertices in $A-a$, that are adjacent, and exactly one vertex in $B-b$,*

($a2.3$) *P' has exactly three components L_1, L_2, L_3 and either each L_i has exactly one vertex in $A-a$ (and, accordingly, has exactly two vertices in $B-b$, that are adjacent) or each L_i has exactly two vertices in $A-a$, that are adjacent (and, accordingly, has exactly one vertex in $B-b$).*

3 Λ-packings in cubic 3-connected graphs

The main goal of this section is to prove the following theorem showing that claim (P) in **1.9** is equivalent to various seemingly stronger claims.

3.1 *The following are equivalent for cubic 3-connected graphs G:*

(**z1**) $v(G) = 0 \bmod 6 \Rightarrow$ *G has a Λ-factor,*

(**z2**) $v(G) = 0 \bmod 6 \Rightarrow$ *for every $e \in E(G)$ there is a Λ-factor of G avoiding e (i.e. $G - e$ has a Λ-factor),*

(**z3**) $v(G) = 0 \bmod 6 \Rightarrow$ *for every $e \in E(G)$ there is a Λ-factor of G containing e,*

(**z4**) $v(G) = 0 \bmod 6 \Rightarrow$ *for every $x \in V(G)$ there is at least one 3-vertex path L such that L is centered at x and $G - L$ has a Λ-factor,*

(**z5**) $v(G) = 0 \bmod 6 \Rightarrow$ *for every $x \in V(G)$ there is at least two 3-vertex paths L such that L is centered at x and $G - L$ has a Λ-factor,*

(**z6**) $v(G) = 0 \bmod 6 \Rightarrow$ *for every $xy \in E(G)$ there are edges $xx', yy' \in E(G)$ such that $G - xyy'$ and $G - x'xy$ have Λ–factors,*

(**z7**) $v(G) = 0 \bmod 6 \Rightarrow$ *$G - X$ has a Λ-factor for every $X \subseteq E(G)$ such that $|X| = 2$,*

(**z8**) $v(G) = 0 \bmod 6 \Rightarrow$ *$G - L$ has a Λ-factor for every 3-vertex path L in G,*

(**z9**) $v(G) = 0 \bmod 6 \Rightarrow$ *for every 3-edge cut K of G and $S \subset K$, $|S| = 2$, there is a Λ-factor P of G such that $E(P) \cap K = S$,*

(**t1**) $v(G) = 2 \bmod 6 \Rightarrow$ *for every $x \in V(G)$ there is $xy \in E(G)$ such that $G - \{x, y\}$ has a Λ-factor,*

(**t2**) $v(G) = 2 \bmod 6 \Rightarrow$ *$G - \{x, y\}$ has a Λ-factor for every $xy \in E(G)$,*

(**t3**) $v(G) = 2 \bmod 6 \Rightarrow$ *for every $x \in V(G)$ there is a 5-vertex path W such that x is the center vertex of W and $G - W$ has a Λ-factor* (see also **2.4** (a1.2) and Fig 3),

(**t4**) $v(G) = 2 \bmod 6 \Rightarrow$ *for every $x \in V(G)$ and $xy \in E(G)$ there is a 5-vertex path W such that x is the center vertex of W, $xy \notin E(W)$, and $G - W$ has a Λ-factor* (see also **2.4** (a1.2) and Fig 3),

(**f1**) $v(G) = 4 \bmod 6 \Rightarrow$ *$G - x$ has a Λ-factor for every $x \in V(G)$,*

(**f2**) $v(G) = 4 \bmod 6 \Rightarrow$ *$G - \{x, e\}$ has a Λ-factor for every $x \in V(G)$ and $e \in E(G)$,*

(**f3**) $v(G) = 4 \bmod 6 \Rightarrow$ *for every $x \in V(G)$ there is a 4-vertex path Z such that x is an inner vertex of Z and $G - Z$ has a Λ-factor* (see also **2.4** (a2.2) and Fig 4)*,*

(**f4**) $v(G) = 4 \bmod 6 \Rightarrow$ *for every $x \in V(G)$ there is $xy \in E(G)$ and a 4-vertex path Z such that x is an inner vertex of Z, $xy \notin E(Z)$, and $G - Z$ has a Λ-factor* (see also **2.4** (a2.2) and Fig 4)*,*

(**f5**) $v(G) = 4 \bmod 6 \Rightarrow$ *for every $xy \in E(G)$ there exists a 4-vertex path Z such that xy is the middle edge of Z and $G - Z$ has a Λ-factor* (see also **2.4** (a2.2) and Fig 4)*,*

(**f6**) $v(G) = 4 \bmod 6 \Rightarrow$ *for every $z \in V(G)$ and every 3-vertex path xyz there exists a 4-vertex path Z such that $xyz \subset Z$, z is an end-vertex of Z, and $G - Z$ has a Λ-factor* (see also **2.4** (a2.2) and Fig 4).

Theorem **3.1** follows from **3.4** – **4.7** below.

In [10] we have shown that claims $(z1)$ - $(z5)$, $(t1)$, $(t2)$, $(f1)$, and $(f2)$ are true for cubic, 3-connected, and claw-free graphs.

The remarks below show that if claims $(z7)$, $(z8)$, $(t2)$, $(f1)$, $(f2)$ in **3.1** are true, then they are best possible in some sense.

$(r1)$ Obviously claim $(z7)$ is not true if condition "$|X| = 2$" is replaced by condition "$|X| = 3$". Namely, if G is a cubic 3-connected graph, $v(G) = 0 \bmod 6$, X is a 3-edge cut in G, and the two components of $G - X$ have different number of vertices mod6, then clearly $G - X$ has no Λ-factor. Also in Section 4 (see **4.5**) we describe an infinite set of cubic 3-connected graphs G having a triangle T such that $G - E(T)$ has no Λ-factor.

$(r2)$ There exist infinitely many triples (G, L, e) such that G is a cubic, 3-connected, bipartite, and planar graph, $v(G) = 0 \bmod 6$, L is a 3-vertex path in G, $e \in E(G - L)$, and $(G - e) - L\}$ has no Λ-factor, and so claim $(z8)$ is tight. Moreover, there are infinitely many triples (G, L, L') such that G is a cubic 3-connected graph, L and L' are disjoint 3-vertex paths in G, and $G - (L \cup L')$ has no Λ-factor. If G has a triangle, then it is easy to find such L and L'. Indeed, let $v \in V(G)$, $N(v, G) = \{x, y, z\}$, and $yz \in E(G)$, and so vyz is a triangle. Since G is 3-connected, x is not adjacent to $\{y, z\}$. Let L and L' be 3-vertex paths in $G - v$ containing x and yz, respectively. Then v is an isolated vertex in $G - (L \cup L')$, and so $G - (L \cup L')$ has no Λ-factor. Similar idea can be used to find such L and L' if G has a 4-cycle. In Section 4 (see **4.6**) we describe a sequence of infinitely many triples (G, L, L') with the above property, where G is a cubic cyclically 6-connected graph, and so G has no triangles, no 4-cycles, and 5-cycles. Thus claim $(z8)$ is tight in this sense as well.

$(r3)$ There exist infinitely many triples (G, xy, e) such that G is a cubic, 3-connected, bipartite, and planar graph, $v(G) = 2 \bmod 6$, $xy \in E(G)$, $e \in E(G - \{x, y\})$, and $G - \{x, y, e\}$ has no Λ-factor, and so claim $(t2)$ is tight.

$(r4)$ There exist infinitely many triples (G, x, y) such that G is a cubic 3-connected graph, $v(G) = 2 \bmod 6$, $\{x, y\} \subset V(G)$, $x \ne y$, $xy \notin E(G)$, and $G - \{x, y\}$ has no Λ-factor, and so claim $(t2)$ is not true if vertices x and y are not adjacent.

$(r5)$ There exist infinitely many (G, a, b, x) such that G is a cubic, 3-connected graph with no 3-cycles and no 4-cycles, $v(G) = 4 \bmod 6$, $x \in V(G)$, a and b are non-adjacent edges in $G - x$, and $G - \{x, a, b\}$ has no Λ-factor, and so claim $(f2)$ is tight.

$(r6)$ There exist infinitely many triples (G, L, x) such that G is a cubic, 3-connected graph with no 3-cycles and no 4-cycles, $v(G) = 4 \bmod 6$, $x \in V(G)$, L is a 3-vertex path in $G - x$, and $G - \{x, L\}$ has no Λ-factor (see claim $(f1)$).

We need the following two results obtained before.

3.2 [7] *Let* $G = Y(A^1, a^1; A^2, a^2; A^3, a^3)$ (see Fig. 2) *and* P *be a* Λ*-factor of* G. *Suppose that each* A^i *is a cubic graph and* $v(A^i) = 0 \bmod 6$. *Then* $cmp(P^i) \in \{1, 2\}$ *for every* $i \in \{1, 2, 3\}$.

Proof Let $i \in \{1, 2, 3\}$. Since D^i is a matching and P^i consists of the components of

P meeting D^i, clearly $cmp(P^i) \leq 3$. Since $v(A^i) = -1 \bmod 6$, we have $cmp(P^i) \geq 1$. It remains to show that $cmp(P^i) \leq 2$. Suppose, on the contrary, that $cmp(P^1) = 3$. Since P is a Λ-factor of G and $v(A^1 - a^1) = -1 \bmod 6$, clearly $v(P^1) \cap V(A^1 - a^1) = 5$ and we can assume (because of symmetry) that P_1 consists of three components $a^1_3 z_3 a^2_3$, $z_1 a^1_1 y^1$, and $z_2 a^1_2 u^1$ for some $y^1, u^1 \in V(A^1)$ Then $cmp(P^3) = 0$, a contradiction. □

3.3 [7] *Let A be a graph, $e = aa_1 \in E(A)$, and $G = Y(A, a)$* (see Fig. 2). *Suppose that*
$(h1)$ A is cubic,
$(h2)$ $v(A) = 0 \bmod 6$, and
$(h3)$ a has no Λ-factor containing $e = aa_1$.

Then $v(G) = 0 \bmod 6$ and G has no Λ-factor.

Proof (uses **3.2**). Suppose, on the contrary, that G has a Λ-factor P. By definition of $G = Y(A, a)$, each A^i is a copy of A and edge $e^i = a^i a^i_1$ in A^i is a copy of edge $e = aa_1$ in A By **3.2**, $cmp(P^i) \in \{1, 2\}$ for every $i \in \{1, 2, 3\}$. Since P is a Λ-factor of G and $v(A^i - x^i) = -1 \bmod 6$, clearly $E(P) \cap D^i$ is an edge subset of a Λ–factor of A^i for every $i \in \{1, 2, 3\}$ (we assume that edge $z_j a^i_j$ in G is edge $a^i a^i_j$ in A^i). Since $a^1 a^i_1$ belongs to no Λ-factor of A^i for every $i \in \{1, 2, 3\}$, clearly $E(P) \cap \{z_1 a^1_1, z_1 a^2_1, z_1 a^3_1\} = \emptyset$. Therefore $z_1 \notin V(P)$, and so P is not a Λ-factor of G, a contradiction. □

3.4 *$(z1) \Leftrightarrow (z2)$.*

Proof (uses **2.3** and **3.2**). Obviously $(z1) \Leftarrow (z2)$. We prove $(z1) \Rightarrow (z2)$.

Suppose, on the contrary, that $(z1)$ is true but $(z2)$ is not true, i.e. there is a cubic 3-connected graph A and $aa_1 \in E(G)$ such that $v(A) = 0 \bmod 6$ and every Λ-factor of G contains aa_1. Let $G = Y(A^1, a^1; A^2, a_2; A^3, a^3)$, where each (A^i, a^i) above is a copy of (A, a) and edge $a^i a^i_1$ in A^i is a copy of edge aa_1 in A (see Fig. 2). Since A is cubic and 3-connected, by **2.3**, G is also cubic and 3-connected. Obviously $v(G) = 0 \bmod 6$. By $(z1)$, G has a Λ-factor P. Since each $v(A^i) = 0 \bmod 6$, by **3.2**, $cmp(P^i) \in \{1, 2\}$ for every $i \in \{1, 2, 3\}$. Since P is a Λ-factor of G and $v(A^i - a^i) = -1 \bmod 6$, clearly $E(P) \cap D^i$ is an edge subset of a Λ-factor of A^i for every $i \in \{1, 2, 3\}$ (we assume that edge $z_j a^i_j$ in G is edge $a^i a^i_j$ in A^i). Since $a^1 a^i_1$ belongs to every Λ-factor of A^i for every $i \in \{1, 2, 3\}$, clearly $\{z_1 a^1_1, z_1 a^2_1, z_1 a^3_1\} \subseteq E(P)$. Therefore vertex z_1 has degree three in P, and so P is not a Λ-factor of G, a contradiction. □

3.5 *$(z1) \Leftrightarrow (z3)$.*

Proof Claim $(z1) \Leftarrow (z3)$ is obvious. Claim $(z1) \Rightarrow (z3)$ follows from **3.3**. □

Let B be a cubic graph. Given $v \in V(B)$, let $A(v)$ be a cubic graph, $a^v \in V(A(v))$, and $A^v = A(v) - a^v$. We assume that all $A(v)$'s are disjoint. Let G be a graph obtained from B by replacing each vertex v in B by A^v. Let $D^v = D(A^v, G)$ and $E' = E(G) \setminus \cup\{E(A^v) : v \in V(B)\}$. Obviously, there is a bijection $\alpha : E(B) \rightarrow E'$ such that if

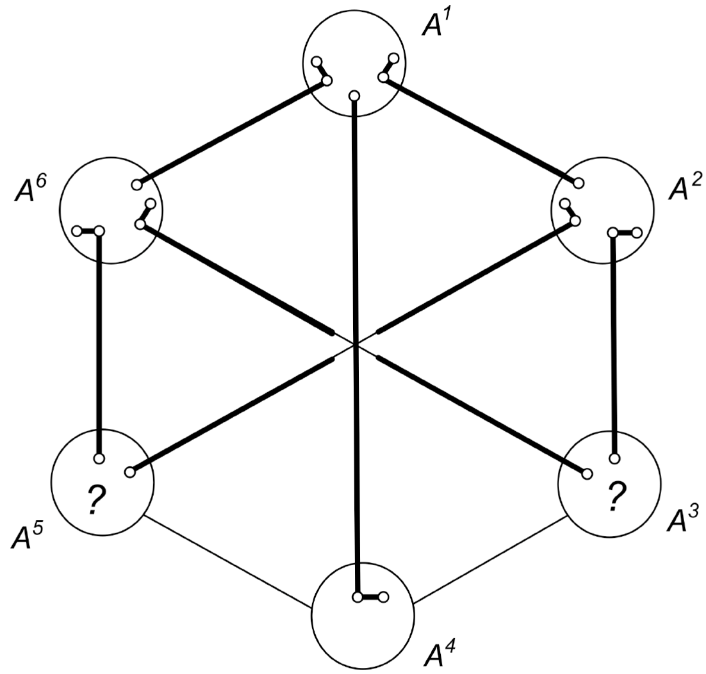

Figure 5: $(z1) \Rightarrow (z4)$

$uv \in E(B)$, then $\alpha(uv)$ is an edge in G having one end-vertex in A^u and the other in A^v.

Let P be a Λ-packing in G. For $uv \in E(B)$, $u \neq v$, we write $u \neg v$ if P has a component L, such that $\alpha(uv) \in E(L)$ and $|V(A^u) \cap V(L)| = 1$. Let P^v be the union of components of P that meet D^v in G.

3.6 $(z1) \Leftrightarrow (z4)$.

Proof (uses **2.2** and **2.4** $(a1)$). Obviously $(z1) \Leftarrow (z4)$. We prove $(z1) \Rightarrow (z4)$.

Suppose that $(z1)$ is true but $(z4)$ is not true, i.e. there is a cubic 3-connected graph A and $a \in V(A)$ such that $v(A) = 0 \bmod 6$ and a has degree one in every Λ-factor of A. Let G be the graph obtained from $B = K_{3,3}$ by replacing each vertex v by a copy A^v of $A - a$ (see Fig. 5). Obviously $v(G) = 0 \bmod 6$ and by **2.2**, G is a cubic, 3-connected graph. By $(z1)$, G has a Λ-factor P. If $uv \in E(P)$, then let $L(uv)$ denote the component of P containing uv. Let $V(B) = \{1, \ldots, 6\}$.

Since vertex a has degree one in every Λ-factor of A and each $(A(v), a^v)$ is a copy of (A, a), by **2.4** $(a1)$, we have: $cmp(P^v) \in \{1, 3\}$.

(**p1**) Suppose that there is $v \in V(B)$ such that $cmp(P^v) = 3$. By symmetry of B, we can assume that $v = 1$ and, by **2.4** $(a1.3)$, $1 \neg 4$, $6 \neg 1$, and $2 \neg 1$ (see Fig. 5). Let $x \in \{2, 6\}$. Since $cmp(P^x) \in \{1, 3\}$ and $|V(L(1x) \cap V(A^x)| = 1$, clearly $cmp(P^x) = 3$ and, by **2.4** $(a1.3)$, $5 \neg x$, $3 \neg x$. Then by **2.4** $(a1.2)$, $cmp(P^s) = 2$ for $s \in \{3, 5\}$, a contradiction.

(**p2**) Now suppose that $cmp(P^v) = 1$ for every $v \in V(B)$. By symmetry, we can assume

$1\neg 2$. Then P^1 contradicts **2.4** ($a1.1$). $\square$

3.7 $(z1) \Leftrightarrow (t1)$.

Proof (uses **2.3**) Let $G = Y(A^1, a^1; A^2, a_2; A^3, a^3)$ (see Fig. 2). By **2.3**, if each A^i is cubic and 3-connected then G is also cubic and 3-connected.

(**p1**) We first prove $(z1) \Rightarrow (t1)$. Suppose, on the contrary, that $(z1)$ is true but $(t1)$ is not true, i.e. there is a cubic 3-connected graph A and $a \in V(A)$ such that $v(A) = 2 \bmod 6$ and $A - \{a, y\}$ has no Λ-factor for every vertex y in A adjacent to a. Let each (A^i, a^i) above be a copy of (A, a), and so $A^i - \{a^i, a^i_j\}$ has no Λ-factor for every $i, j \in \{1, 2, 3\}$. Obviously $v(G) = 0 \bmod 6$. By $(z1)$, G has a Λ-factor P. Then it is easy to see that since each $v(A^i - a^i) = 1 \bmod 6$, there are $r, s, j \in \{1, 2, 3\}$ such that $r \neq s$ and $P^s = x^s_j z_j x^r_j$. Since P is a Λ-factor of G, clearly $P \cap (A^s - \{z_j, a^s_j\})$ is a Λ-factor of $A^s - \{z_j, a^s\} = A^s - \{a^s, x^s_j\}$, a contradiction.

(**p2**) Now we prove $(z1) \Leftarrow (t1)$. Suppose, on the contrary, that $(t1)$ is true but $(z1)$ is not true, i.e. there is a cubic 3-connected graph A such that $v(A) = 0 \bmod 6$ and A has no Λ-factor.

Let (A^i, a^i) above be a copy of (A, a) for $i \in \{1, 2\}$, where $a \in V(A)$, and (A^3, a^3) be a copy of (H, h) for some cubic 3-connected graph H and $h \in V(H)$, where $v(H) = 2 \bmod 6$. Obviously $v(G) = 2 \bmod 6$. Suppose that P is a Λ-factor of $G - \{z_1 a^3_1\}$. Then $cmp(P^1) \leq 2$. Since $v(A^1 - a^1) = -1 \bmod 6$, we have $cmp(P^1) \geq 1$. Now since P is a Λ–factor of $G - \{z_1 a^3_1\}$ and $v(A^1 - a^1) = -1 \bmod 6$, clearly $E(P) \cap D^1$ is an edge subset of a component of a Λ-factor of A^1 (we assume that edge $z_j a^1_j$ in G is edge $a^1 a^1_j$ in A^1). Therefore A has a Λ-factor, a contradiction. $\square$

3.8 $(z1) \Leftrightarrow (f1)$.

Proof (uses **2.3**) Let $G = Y(A^1, a^1; A^2, a_2; A^3, a^3)$ (see Fig. 2). By **2.3**, if each A^i is cubic and 3-connected then G is also cubic and 3-connected.

(**p1**) We first prove $(z1) \Rightarrow (f1)$. Suppose, on the contrary, that $(z1)$ is true but $(f1)$ is not true, i.e. there is a cubic 3-connected graph A such that $v(A) = 4 \bmod 6$ but $A - a$ has no Λ-factor for some $a \in V(A)$. Let each (A^i, a^i) above be a copy of (A, a). Obviously $v(G) = 0 \bmod 6$. By $(z1)$, G has a Λ-factor P. Let $E^i = E^i(P) = D^i \cap E(P)$.

Since P is a Λ-factor of G and $v(A^i - a^i) = 3 \bmod 6$, clearly $|E^i| \in \{0, 2, 3\}$. Since $A^i - a^i$ has no Λ-factor, $|E^i| \in \{2, 3\}$ for $i \in \{1, 2, 3\}$. Then each $d(z_j, P) \geq 2$. Since P is a Λ-factor of G, clearly each $d(z_j, P) \leq 2$. Therefore each $d(z_j, P) = 2$. But then $|E^i| = 1$ for some $i \in \{1, 2, 3\}$, a contradiction.

(**p2**) Now we prove $(z1) \Leftarrow (f1)$. Suppose, on the contrary, that $(f1)$ is true but $(z1)$ is not true, i.e. there is a cubic 3-connected graph A such that $v(A) = 0 \bmod 6$ and A has no Λ-factor. Let (A^i, a^i) above be a copy of (A, a) for $i \in \{1, 2\}$ and some $a \in V(A)$ and (A^3, a^3) is a copy of (H, h) for some cubic 3-connected graph H and $h \in V(H)$, where $v(H) = 4 \bmod 6$ (see Fig. 2). Then $v(G) = 4 \bmod 6$. Let $x \in V(H - h)$. Suppose that

$G - x$ has a Λ-factor P. Since A has no Λ-factor, clearly $|E^1(P)| = |E^2(P)| = 3$. Then P is not a Λ-factor of $G - x$, and so $(f1)$ is not true, a contradiction. $\square$

3.9 $(z4) \Leftrightarrow (t2)$.

Proof (uses **2.3**, **3.6**, and **3.7**). Let $G = Y(A^1, a^1; A^2, a_2; A^3, a^3)$ (see Fig. 2). By **2.3**, if each A^i is cubic and 3-connected, then G is also cubic and 3-connected.

(**p1**) We first prove $(z4) \Leftarrow (t2)$. Obviously $(t2) \Rightarrow (t1)$. By **3.6**, $(z1) \Leftrightarrow (z4)$. By **3.7**, $(z1) \Leftrightarrow (t1)$. The result follows.

(**p2**) Now we prove $(z4) \Rightarrow (t2)$. Suppose, on the contrary, that $(z4)$ is true but $(t2)$ is not true. Then there is a cubic 3-connected graph A and $aa_1 \in E(G)$ such that $v(A) = 2 \bmod 6$ and $A - \{a, a_1\}$ has no Λ-factor. Let each (A^i, a^i) above be a copy of (A, a) and edge $a^i a^i_1$ in A^i be a copy of edge aa_1 in G. Obviously $v(G) = 0 \bmod 6$. Let $L^i = a^j_1 z_1 a^k_1$, where $\{i, j, k\} = \{1, 2, 3\}$. By $(z4)$, G has a Λ-factor P containing L^i for some $i \in \{1, 2, 3\}$, say for $i = 3$. If $s \in \{1, 2\}$, then $cmp(P^s) = 3$ because $A_s - \{a^s, a^s_1\}$ has no Λ-factor and $v(A^s - a^s) = 1 \bmod 6$. Also $cmp(P^3) \ge 1$ because $v(A^3 - a^3) = 1 \bmod 6$. Then $P_1 \cup P^2 \cup P^3$ has at least four components each meeting $\{z_1, z_2, z_3\}$, a contradiction. $\square$

3.10 $(z2) \Rightarrow (z5)$.

Proof (uses **2.1** and **2.4** $(a1)$). Suppose, on the contrary, that $(z2)$ is true but $(z5)$ is not true. Then there is a cubic 3-connected graph A and $a \in V(A)$ such that at most one 3-vertex path, centered at a and belonging to a Λ-factor of A. It is sufficient to prove our claim in case when A has exactly one 3-vertex path, say $L = a_1 a a_2$, centered at a and belonging to a Λ-factor of A. Let $e_i = aa_i$, and so $E(L) = \{e_1, e_2\}$.

Let B be the graph-skeleton of the three-prism, say, $V(B) = \{1, \ldots, 6\}$ and B is obtained from two disjoint triangles 123 and 456 by adding three new edges 14, 25, and 36.

Let each $(A(v), a^v, a^v_1, a^v_1)$, $v \in V(B)$ be a copy of (A, a, a_1, a_2), and so edge $e^v_i = a^v a^v_i$ in $A(v)$ is a copy of edge $e_i = aa_i$ in A, $i \in \{1, 2\}$. We also assume that all $A(v)$'s are disjoint. Let G be a graph obtained from B by replacing each $v \in V(B)$ by $A^v = A(v) - a^v$) (see Fig. 6). Given $v \in V(B)$, let $S(v)$ be the set of two edges e'_i in E' such that edge e'_i is incident to vertex a^v_i in G, $\{i \in \{1, 2\}$. We assume that each vertex v in B is replaced by A^v (to obtain G) in such a way that

$S(x_1) = \{\alpha(13), \alpha(1\}$, $S(x_2) = \{\alpha(21), \alpha(23)\}$, $S(x_3) = \{\alpha(32), \alpha(36)\}$,

$S(y_1) = \{\alpha(45), \alpha(46)\}$, $S(y_2) = \{\alpha(54), \alpha(56)\}$, $S(y_3) = \{\alpha(63), \alpha(64)\}$.

In Figure 6 the edges in $S(v)$ are marked for every $v \in V(B)$.

By **2.1**, G is a cubic, 3–connected graph. Since $v(B) = 0 \bmod 6$, clearly also $v(G) = 0 \bmod 6$. By $(z2)$, $G' = G - \alpha(36)$ has a Λ-factor, say P.

We know that A has exactly one 3-vertex path $L = a_1 a a_2$ centered at a and belonging to a Λ-factor of A and that each (A^v, a^v, a^v_1, a^v_2) is a copy of (A, a, a_1, a_2), and so $v(A^v -$

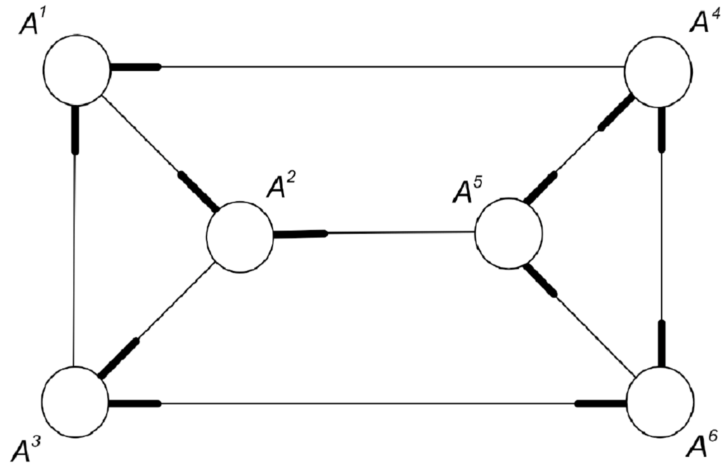

Figure 6: $(z2) \Rightarrow (z5)$

$a^v) = 1 \bmod 6$. Therefore by **2.4** $(a1)$ the Λ-factor P satisfies the following condition for every $v \in V(B)$:

c(**v**) if $cmp(P^v) = 2$ then $v\neg a$ and $v\neg b$, where $\{\alpha(va), \alpha(vb)\} = S(v)$.

Obviously $|D^3| = |D^6| = 2$ in $G - \alpha(36)$. Therefore $cmp(P^3) \leq 2$ and $cmp(P^6) \leq 2$. Since $\alpha(36) \in S(3) \cap S(6)$, by conditions **c**(**3**) and **c**(**6**), $cmp(P^3) = cmp(P^6) = 1$. Now by **2.4** $(a1)$, $x'\neg 3$ for some $x' \in \{1, 2\}$ and $y'\neg 6$ for some $y' \in \{4, 5\}$.

(**p1**) Suppose that $1\neg 3$. Assume first that $\alpha(14) \notin E(P)$. Then $cmp(P^1) \leq 2$. By **2.4** $(a1)$, $cmp(P^1) = 2$. This contradicts **c**(**1**). Thus we can assume that $\alpha(14) \in E(P)$.

(**p1.1**) Suppose that $4\neg 6$.
Suppose that $1\neg 4$. Then by **2.4** $(a1)$, $5\neg 4$ and $5\neg 2$. This contradicts **c**(**5**).
Now suppose that $4\neg 1$. This contradicts **c**(**4**).

(**p1.2**) Suppose that $5\neg 6$. Then $cmp(P^4) \leq 2$.
Suppose that $1\neg 4$. By **2.4** $(a1.1)$, $cmp(P^4) = 1$. Then $5\neg 2$. This contradicts **c**(**5**).
Now suppose that $4\neg 1$. Then $cmp(P^4) = 2$. This contradicts **c**(**4**).

(**p2**) Now suppose that $2\neg 3$. Then $cmp(P^1) \leq 2$. By **c**(**2**) and **2.4** $(a1.3)$, $x_1\neg x_2$ (and $y_2\neg x_2$). Then by **2.4** $(a1.2)$, $cmp(P^1) = 2$. This contradicts **c**(**1**). □

3.11 $(z7) \Rightarrow (z5)$.

A proof of **3.11** can be obtained from the above **Proof** of **3.10** by using $(z7)$ instead of $(z2)$ and by eliminating (**p.1.1**).

3.12 $(z1) \Leftrightarrow (z6)$.

Proof Obviously $(z1) \Leftarrow (z6)$ and $(z5) \Rightarrow (z6)$. Now $(z1) \Rightarrow (z6)$ follows from **3.4** and **3.10**. □

3.13 $(z7) \Rightarrow (z8)$.

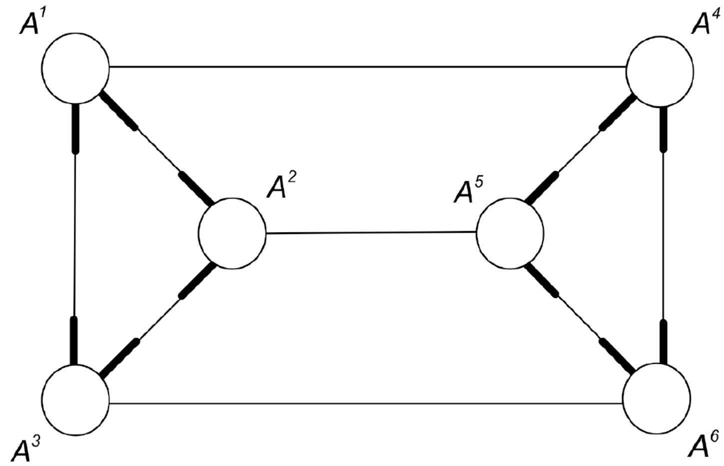

Figure 7: $(z7) \Rightarrow (z8)$

Proof 1. Suppose, on the contrary, that $(z7)$ is true but $(z8)$ is not true. Then there is a cubic 3-connected graph A and a 3-path $L = a_1aa_2$ in A such that $v(A) = 0 \bmod 6$ and $A - L$ has no Λ-factor. Let $N(a, A) = \{a_1, a_2, a_3\}$. Let $(A^i; a^i, a^i_1, a_2, a^i_3)$, $i \in \{1, 2\})$, be two copies of $(A; a, a_1, a_2, a_3)$ and A^1, A^2 be disjoint graphs, and so $L^i = a^i_1a^ia^i_2$ in A^i is a copy of $L = a_1aa_2$ in A. Let $G = A^1a^1\sigma a^2A^2$, where $\sigma : N(a^1, A^1) \to N(a^2, A^2)$ is a bijection such that $\sigma(a^1_i) = a^2_i$ for $i \in \{1, 2, 3\}$. Let H be the graph obtained from G by subdividing edge $a^1_ja^2_i$ by a new vertex v_j for $j \in \{1, 2\}$ and by adding a new edge v_1v_2 Obviously G is a cubic 3-connected graph and $v(G) = 0 \bmod 6$. By $(z7)$, $G - \{v_1v_2, a^1_3a^2_3\}$ has a Λ-factor, say P. Since $v(A^i - a^i) = -1 \bmod 6$, clearly $a^1_1v_1a^2_1$ and $a^1_2v_1a^2_2$ are components of P. Then $A - \{a_1, a_2\} = A - L$ has a Λ-factor, a contradiction. $\square$

Proof 2 (uses **2.4** $(a1)$). Suppose, on the contrary, that $(z7)$ is true but $(z8)$ is not true. Then there is a cubic 3-connected graph A and a 3-path $L = a_1aa_2$ in A such that $v(A) = 0 \bmod 6$ and $A - L$ has no Λ-factor. Let $N(a, A) = \{a_1, a_2, a_3\}$.

Let B, $\{(A(v), a^v, a^v_1, a^v_2, a^v_3) : v \in V(B)\}$, and G be as in **3.10** (see Fig. 7).

Given $v \in V(B)$, let $S(v)$ be the set of two edges e'_i in E' such that edge e'_i is incident to vertex a^v_i in G, $\{1 \in \{1, 2\}$. We assume that each vertex v in B is replaced by A^v (to obtain G) in such a way that

$S(x_1) = \{\alpha(12), \alpha(13)\}$, $S(x_2) = \{\alpha(21), \alpha(23)\}$, $S(x_3) = \{\alpha(32), \alpha(31)\}$,
$S(y_1) = \{\alpha(45), \alpha(46)\}$, $S(y_2) = \{\alpha(54), \alpha(56)\}$, $S(y_3) = \{\alpha(64), \alpha(65)\}$.
In Figure 7 the edges in $S(v)$ are marked for every $v \in V(B)$.

Since $v(G) = 0 \bmod 6$ and G is cubic and 3-connected, by $(z7)$, $G - \{\alpha(14), \alpha(3, 6)\}$ has a Λ-factor, say P. By **2.4** $(a1)$, for every $v \in V(B)$, the Λ-factor P satisfies the following condition:

$\mathbf{c(v)}$ if $cmp(P^v) = 2$ then $v\neg a$ and $v\neg b$ where $\{\alpha(va), \alpha(vb)\} \neq S(v)$.

Obviously $|D^i| = 2$ in $G - \{\alpha(14), \alpha(3,6)\}$, and so $cmp(P^i) \le 2$ for $i \in \{1,3\}$. Since $S(1) = \{12, 13\}$ and $S(3) = \{31, 32\}$, by conditions $\mathbf{c(1)}$ and $\mathbf{c(4)}$ we have: $cmp(P^1 = cmp(P^3) = 1$. Now by **2.4** $(a1.1)$, $2\neg 1$ and $2\neg 3$. This contradicts $\mathbf{c(1)}$. $\square$

3.14 $(z8) \Rightarrow (z7)$.

Proof Obviously $(z8) \Rightarrow (z4)$. By **3.9**, $(z4) \Rightarrow (t2)$ and by **3.15**, $(t2) \Rightarrow (z7)$. $\square$

3.15 $(t2) \Leftrightarrow (z7)$.

Proof (uses **3.6** and **3.9**). We first prove $(t2) \Rightarrow (z7)$. Let G be a cubic, 3-connected graph with $v(G) = 0 \bmod 6$ and $a = a_1a_2$, $b = b_1b_2$ be two distinct edges of G. Let G' be the graph obtained from G as follows: subdivide edge a_1a_2 by a new vertex a' and edge b_1b_2 by a new vertex b' and add a new edge $e = a'b'$. Then G' is a cubic and 3-connected graph, $v(G') = 2 \bmod 6$, and $G - \{a, b\} = G' - \{a', b'\}$. By $(t2)$, $G' - \{a', b'\}$ has a Λ-factor.

Now we prove $(t2) \Leftarrow (z7)$. Obviously $(z7) \Rightarrow (z1)$. By **3.6**, $(z1) \Rightarrow (z4)$ and by **3.9**, $(z4) \Rightarrow (t2)$. Implication $(t2) \Leftarrow (z7)$ also follows from obvious $(z8) \Rightarrow (z4)$, from $(z7) \Rightarrow (z8)$, (by **3.13**), and from $(z4) \Rightarrow (t2)$ (by **3.9**). $\square$

Here is a direct proof of $(z7) \Rightarrow (t2)$.

3.16 $(z7) \Rightarrow (t2)$.

Proof Let G be a cubic, 3-connected graph, $v(G) = 2 \bmod 6$, $xy \in E(G)$, $N(x, G) = \{x_1, x_2, y\}$, and $N(y, G) = \{y_1, y_2, x\}$. Let $G_1 = G - \{x, y\} \cup E_1$, $G_2 = G - \{x, y\} \cup$, and $G_3 = G - \{x, y\} \cup E_3$, where $E_1 = \{x_1y_1, x_2y_2\}$, $E_2 = \{x_1y_2, x_2y_1\}$, and $E_3 = \{x_1x_2, y_1y_2\}$. Obviously each G_i is a cubic graph. It is easy to see that since G is 3-connected, there is $s \in \{1, 2, 3\}$ such that G_s is 3–connected. Clearly $G_s - E_s = G - \{x, y\}$. By $(z7)$, $G_s - E_s$ has a Λ-factor. $\square$

3.17 $(z1) \Leftrightarrow (z8)$.

Proof Obviously $(z8) \Rightarrow (z1)$. By **3.6**, $(z1) \Rightarrow (z4)$. By **3.9**, $(z4) \Rightarrow (t2)$. By **3.15**, $(t2) \Rightarrow (z7)$. By **3.13**, $(z7) \Rightarrow (z8)$. Therefore $(z1) \Rightarrow (z8)$. $\square$

3.18 $(z8) \Rightarrow (f1)$.

Proof Let G be a cubic 3-connected graph, $v(G) = 4 \bmod 6$, $x \in V(G)$, and $N(x, G) = \{x_1, x_2, x_3\}$. Let G' be the graph obtained from G by replacing x by a triangle T with $V(T) = \{x'_1, x'_2, x'_3\}$ so that $x_ix'_i \in E(G')$, $i \in \{1, 2, 3\}$. Since $v(G) = 4 \bmod 6$, clearly $v(G') = 0 \bmod 6$. Consider the 3-vertex path $L' = x'_1x'_2x'_3$ in G'. By $(z8)$, $G' - L'$ has a Λ-factor, say P'. Obviously $P' - L'$ is a Λ-factor of $G' - L'$ and $G - x = G' - L'$. $\square$

3.19 $(z8) \Rightarrow (f2)$.

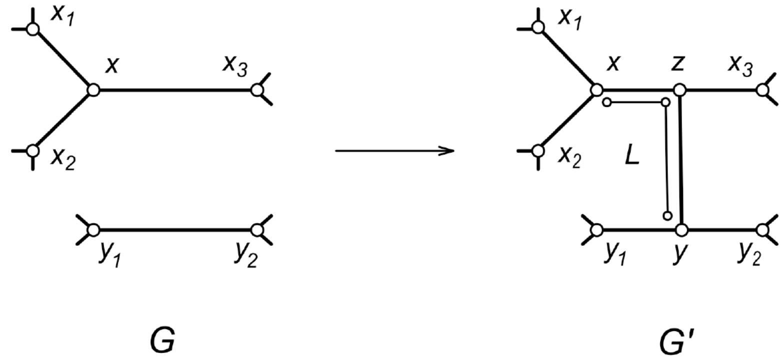

Figure 8: $(z8) \Rightarrow (f2)$

Proof (uses **3.18**). Let G be a cubic 3-connected graph, $v(G) = 4 \bmod 6$, $x \in V(G)$, and $e = y_1y_2 \in E(G)$. We want to prove that if $(z8)$ is true, then $G - \{x, e\}$ has a Λ-factor. If $x \in \{y_1, y_2\}$, then $G - \{x, e\} = G - x$, and therefore by **3.18**, our claim is true. So we assume that $x \in \{y_1, y_2\}$. Let $N(x, G) = \{x_1, x_2, x_3\}$. Let G' be the graph obtained from G by subdividing edge y_1y_2 by a vertex y and edge xx_3 by a vertex z and by adding a new edge yz (see Fig. 8). Since $v(G) = 4 \bmod 6$, clearly $v(G') = 0 \bmod 6$. Since $x \notin \{y_1, y_2\}$ and G is cubic and 3-connected, G' is also cubic and 3-connected. Obviously $L = xzy$ is a 3-vertex path in G' and $G - \{x, e\} = G - \{x, y_1y_2\} = G' - L$. By $(z8)$, $G' - L$ has a Λ-factor. $\square$

3.20 $(f2) \Rightarrow (t4)$.

Proof Let G be a cubic, 3-connected graph, $v(G) = 2 \bmod 6$, $x \in V(G)$, and $N(x, G) = \{x_1, x_2, x_3\}$. Let G' be the graph obtained from G by replacing x by a triangle T with $V(T) = \{x'_1, x'_2, x'_3\}$ so that $x_ix'_i \in E(G')$, $i \in \{1, 2, 3\}$. Since $v(G) = 2 \bmod 6$, clearly $v(G') = 4 \bmod 6$. By $(f2)$, $G' - \{x'_i, x'_jx'_k\}$ has a Λ-factor, say P_i where $\{i, j, k\} = \{1, 2, 3\}$. Since $x_jx'_j$ and $x_kx'_k$ are dangling edges in $G' - \{x'_i, x'_jx'_k\}$, clearly $x_jx'_j, x_kx'_k \in E(P_i$ and $d(x'_j, P_i) = d(x'_k, P_i) = 1$. Let L_j and L_k be the components of P_i containing $x_jx'_j$ and $x_kx'_k$, respectively. Then $E(L_j) \cup E(L_k)$ induces in G a 5-vertex path W_i such that x is the center vertex of W_i and $x_ix'_i \notin E(W_i)$. $\square$

3.21 $(t3) \Rightarrow (z1)$.

Proof (**2.4** $(a1.3)$). Let $G = Y(A^1, a^1; A^2, a_2; A^3, a^3)$ (see Fig. 2). Suppose, on the contrary, that $(t3)$ is true but $(z1)$ is not true, i.e. there is a cubic 3-connected graph A such that $v(A) = 0 \bmod 6$ and A has no Λ-factor. Let $a \in V(A)$. Let (A^i, a^i) above be a copy of (A, a) for $i \in \{1, 2\}$ and let (A^3, a^3) be such that $v(A_3) = 2 \bmod 6$. Obviously $v(G) = 2 \bmod 6$. By $(t3)$, G has a 5-vertex path W such that z_3 is the center vertex of W and $G - W$ has a Λ-factor, say P. Obviously $(A^i - a^i) \cap W = \emptyset$ for some $i \in \{1, 2, 3\}$.

Suppose that $(A^3 - a^3) \cap W = \emptyset$. Then W has an end-edge in $A^1 - a^1$ and in $A^2 - a^2$. Since A has no Λ-factor, by **2.4** $(a1.3)$, $D^i - e(W) \subseteq E(P)$ for $i \in \{1, 2\}$. Then P is not a Λ-factor of $G - W$, a contradiction.

Now suppose that $(A^3 - a^3) \cap W \neq \emptyset$. By symmetry, we can assume that $(A^2 - a^2) \cap W = \emptyset$. Then W has an end–edge in $A^1 - a^1$ and in $A^3 - a^3$. Then by **2.4** (a1.3), $Cmp(P^1) = \{L_1, L_2\}$, where L_1 has an end-vertex in $A^1 - a^1$ and L_2 has an end-edge in $A^1 - a^1$. By symmetry, we can assume that $a^1_i z_i \in E(L_i)$ for $i \in \{1, 2\}$. Then $L_1 = a^1_1 z_1 y$, where $y \in \{a^2_1, a^3_1\}$ and z_2 is of degree one in P. Then P is not a Λ-factor of $G - W$, a contradiction. □

3.22 $(z8) \Rightarrow (f4)$.

Proof Let G be a cubic, 3-connected graph, $v(G) = 4 \bmod 6$, $x \in V(G)$, and $N(x, G) = \{x_1, x_2, x_3\}$. Let G' be the graph obtained from G by replacing x by a triangle Δ with $V(\Delta) = \{x'_1, x'_2, x'_3\}$ so that $x_i x'_i \in E(G')$, $i \in \{1, 2, 3\}$. Since $v(G) = 4 \bmod 6$, clearly $v(G') = 0 \bmod 6$. Consider the 3-vertex path $L'_i = x_j x'_j x'_k$ in G', where $\{i, j, k\} = \{1, 2, 3\}$. By $(z8)$, $G' - L'_i$ has a Λ-factor, say P_i. Since $x_i x'_i$ is a dangling edge in $G' - L'_i$, clearly $x_i x'_i \in E(P_i$ and $d(x'_i, P_i) = 1$. Let L_i be the components of P_i containing $x_i x'_i$. Then $E(L_i) \cup x_j x'_j$ induces in G a 4-vertex path Z_k such that x_i is an inner vertex of Z_k and $x_k x'_k \notin E(Z_k)$. □

Let H' be a tree such that $V(H') = \{x, y\} \cup (b^j : j \in \{1, 2, 3, 4\}$ and $E(G) = \{xy, b^1 x, b^2 x, b^3 y, b^4 y\}$. Let H_i, $i \in \{1, 2, 3\}$ be three disjoint copies of H' with $V(H_i) = \{x_i, y_i\} \cup (b^j_i : j \in \{1, 2, 3, 4\}$. Let H be obtained from these three copies by identifying for every j three vertices b^j_1, b^j_2, b^j_2 with a new vertex z^j. Let A^i, $i \in \{1, 2, 3, 4\}$, be a cubic graph, $a^i \in V(A^i)$ and let $G = H(A^1, a^1; A^2, a_2; A^3, a^3; A^4, a^4)$ be the graph obtained from H by replacing each z^j by $A^j - a^j$ assuming that all A^i's are disjoint,

3.23 $(f3) \Rightarrow (z1)$.

Proof (uses **2.2**). Let $G = H(A^1, a^1; A^2, a_2; A^3, a^3; A^4, a^4)$, where each A^i is a cubic 3-connected graph. Since H is cubic and 3-connected, by **2.2**, G is also cubic and 3-connected.

Suppose, on the contrary, that $(f3)$ is true but $(z1)$ is not true, i.e. there is a cubic 3-connected graph A such that $v(A) = 0 \bmod 6$ and A has no Λ-factor. Let $a \in V(A)$. Let (A^i, a^i) be a copy of (A, a) for $i \in \{1, 2, 3\}$ and let $v(A^4) = 2 \bmod 6$. Obviously $v(G) = 4 \bmod 6$. It is easy to see that $G - Z$ has no Λ-factor for every 4-vertex path Z in G such that y_1 is an inner vertex of Z. This contradicts $(f3)$. □

Obviously $(f4) \Rightarrow (f3)$. Therefore from **3.23** we have: $(f4) \Rightarrow (z1)$. Below we give a direct proof of this implication.

3.24 $(f4) \Rightarrow (z1)$.

Proof Let $G = Y(A^1, a^1; A^2, a_2; A^3, a^3)$ (see Fig. 2). Suppose, on the contrary, that $(f4)$ is true but $(z1)$ is not true, i.e. there is a cubic 3-connected graph A such that $v(A) = 0 \bmod 6$ and A has no Λ-factor. Let $a \in V(A)$. Let (A^3, a^3) above be a copy of (A, a) and let (A^i, a^i) for $i \in \{1, 2\}$ be copies of (B, b) where B is a cubic 3-connected

graph, $v(B) = 2 \bmod 6$, and $b \in V(B)$. Obviously $v(G) = 4 \bmod 6$. By $(f4)$, G has a 4-vertex path Z such that z_3 is an inner vertex of Z, $(A^3 - a^3) \cap Z = \emptyset$, and $G - Z$ has a Λ-factor, say P. Since A^3 has no Λ-factor, clearly P is not a Λ-factor of $G - Z$, a contradiction. $\square$

Implication $(z8) \Rightarrow (f1)$ follows from obvious $(z8) \Rightarrow (z1)$ and from $(z1) \Rightarrow (f1)$, by **3.8**. It also follows from obvious $(f2) \Rightarrow (f1)$ and from $(z8) \Rightarrow (f2)$, by **3.19**. Below we give a direct proof of this implication.

3.25 $(f6) \Rightarrow (f5) \Rightarrow (f4) \Rightarrow (z1) \Rightarrow (f6)$.

Proof (uses **3.24** and **3.17**). Obviously $(f6) \Rightarrow (f5) \Rightarrow (f4)$. By **3.24**, $(f4) \Rightarrow (z1)$. Therefore $(f6) \Rightarrow (z1)$. It remains to prove $(z1) \Rightarrow (f6)$. By **3.17**, $(z1) \Rightarrow (z8)$. Thus it is sufficient to show that $(z8) \Rightarrow (f6)$. Let G be a cubic 3-connected graph, $v(G) = 4 \bmod 6$, and xyz is a 3-vertex path in G. Let $N(y, G) = \{x, z, s\}$ and G' be obtained from G by subdividing edges yz and ys by new vertices z' and s', respectively, and by adding a new edge $s'z'$. Then G' is a cubic 3-connected graph and $v(G') = 0 \bmod 6$. Consider the 3-vertex path $L' = zz's'$ in G'. By $(z8)$, G' has a Λ-factor P' containing L'. Since vertex y has degree one in $G' - L'$, clearly P' has a 3-vertex path $Q' = yxq$. Let Z be the 4-vertex path $qxyz$ in G. Then $P' - (P' \cup Q')$ is a Λ-factor in $G - Z$. $\square$

3.26 $(z9) \Leftrightarrow (z1)$.

Proof Obviously $(z9) \Rightarrow (z8)$. By the above claims, $(z1)$, $(z8)$, $(t4)$, and $(f6)$ are equivalent. So we can use these claims to prove $(z9)$. Let G be a cubic 3-connected graph, K a 3-edge cut of G, $S \subset K$ and $|S| = 2$, and $v(G) = 0 \bmod 6$. If the edges of K are incident to the same vertex x in G (i.e. $K = D(x, G)$), then by $(z8)$, G has a Λ-factor P of G such that $E(P) \cap K = S$, and so our claim is true. So we assume that the edges in K are not incident to the same vertex in G. Then since G is cubic and 3-connected, clearly 3-edge cut K is matching. Let A and B be the two component of $G - K$. By the above arguments, we assume that $v(A) \neq 1$ and $v(B) \neq 1$. Let A^b be the graph obtained from G by identifying the vertices of B with a new vertex b and similarly, B^a be the graph obtained from G by identifying the vertices of A with a new vertex a, and so $A = A^b - b$ and $B = B_a - a$. Then $D(b, A_b) = D(a, B_a) = D$, and $S \subset D$. Let $S = \{a_1b_1, a_2b_2\}$, where $\{a_1, a_2\} \subset V(A)$ and $\{b_1, b_2\} \subset V(B)$. Obviously S forms a 3-vertex path $S_A = a_1ba_2$ in A^b and a 3-vertex path $S_B = b_1ab_2$ in B^a. Since G is cubic and 3-connected, both A^b and B^a are also cubic and 3-connected. Since $v(G) = 0 \bmod 6$, there are two possibilities (up to symmetry):
$(c1)$ $v(A^b) = 0 \bmod 6$ and $v(B^a) = 2 \bmod 6$ and
$(c2)$ $v(A^b) = v(B^a) = 4 \bmod 6$.

Consider case $(c1)$. By $(z8)$, $A^b - S_A$ has a Λ-factor P_A. By $(t4)$, B^a has a 5-vertex path W such a is the center vertex of W, $S_B \subset W$, and $B^a - W$ has a Λ-factor P_B.

Then $E(P_A) \cup E(P_B) \cup S$ induces a Λ-factor P in G such that $E(P) \cap K = S$.

Now consider case $(c2)$. By $(f6)$, we have:

(a) A^b has a 4-vertex path Z_A such that a_1 is an end-vertex of Z_A, $S_A \subset Z_A$, and $A^b - Z_A$ has a Λ-factor, say P_A, and similarly,

(b) B^a has a 4-vertex path Z_B such that b_2 is an end-vertex of Z_B, $S_B \subset Z_B$, and $B^a - Z_B$ has a Λ-factor, say P_B.

Then $E(P_A) \cup E(P_B) \cup S$ induces a Λ-factor P in G such that $E(P) \cap K = S$. $\square$

4 On almost cubic graphs with no Λ-factors

In the previous section we indicated that some claims in **3.1** (equivalent to $(z1)$) are best possible in some sense. In this section we describe constructions that provide some additional facts of this nature.

Let $G = Y(A^1, a^1; A^2, a^2; A^3, a^3)$ (see Fig. 2), where A^3 is the graph having two vertices x, a^3 and three parallel edges with the end-vertices x, a^3, and so $A^3 - a^3 = x$.

If $v(A^1) = v(A^2) = 0 \bmod 6$, then

$(a1)$ $v(G) = 2 \bmod 6$ and $G - (N(x, G) \cup x \cup y)$ has no Λ-factor for every vertex y in $G - (N(x, G) \cup x)$ adjacent to a vertex in $N(x, G)$ (see also **2.4** $(a1.3)$ and Fig 3).

If $v(A^1) = 2$ and $v(A^2) = 4 \bmod 6$, then

$(a2)$ $v(G) = 4 \bmod 6$ and $G - (x \cup N(x))$ has no Λ-factor (see also **2.4** $(a2.3)$ and Fig 4).

Thus from the above construction we have:

4.1 *There are infinitely many pairs (G, x) such that G is a cubic 3-connected graph, $x \in V(G)$, and (G, x) satisfies (aj) above, $j \in \{1, 2\}$.*

Using **4.1**, one can also prove the following.

4.2 *There are infinitely many cubic 3-connected graphs G such that $v(G) = 0 \bmod 6$ and $|E(P) \cap K| \in \{1, 2\}$ for every Λ-factor P of G and every 3-edge cut K of G.*

Now we want to define the class $\mathcal{F}$ of graphs G that have some special Λ-packing properties and that are 'almost' cubic. Using these graphs we will construct cubic 3-connected graphs mentioned in our above remark $(r1)$ concerning the result in **3.1**.

Let $L(G)$ denote the set of leaves (i.e. of vertices having degree one) of a graph G. If T is a subgraph of G, then let $N(T, G)$ be the set of vertices in G adjacent to some vertices in T and, as above, $D(T, G)$ the set of edges in G having exactly one end in T.

First we define two special graphs Y and Z. Let Y be the graph obtained from a triangle T with $V(T) = \{z_1, z_2, z_3\}$ by adding three new vertices x_1, x_2, x_3 and three new edges x_1y_1, x_2y_2, x_3y_3), and so x_1, x_2, x_3 are the leaves of Y_0 (see Fig 9. Let Y'

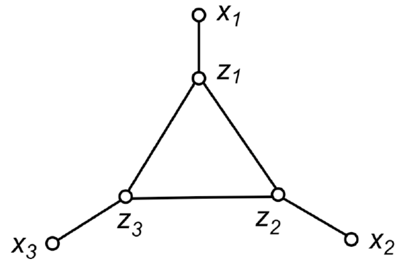

Figure 9: Graph Y

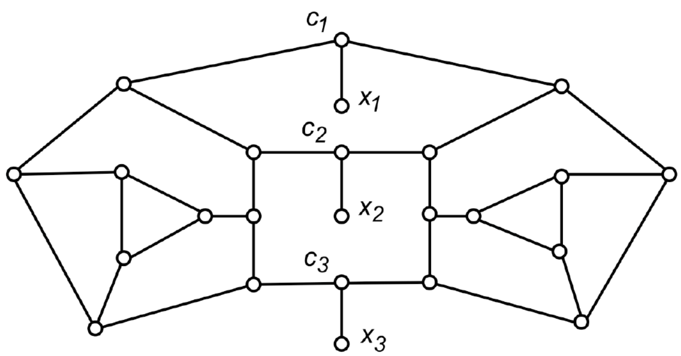

Figure 10: Graph Z

be a copy of Y with the leaves y_1, y_2, y_3. Let S be the graph obtained from Y' by adding six new vertices $\{s_i, s'_i : i \in \{1,2,3\}\}$ and nine new edges $\{s'_i y_j, s'_i y_k : \{i,j,k\} = \{1,2,3\}\} \cup \{s_i s'_i : i \in \{1,2,3\}\}$, and so s_1, s_2, s_3 are the leaves of S. Let A and B be two disjoint copies of S with the leaves a_1, a_2, a_3 and b_1, b_2, b_3, respectively. Let Z be the graph obtained from A and B by identifying a_i and b_i with a new vertex c_i, $i \in \{1,2,3\}$, and by adding three new vertices x_1, x_2, x_3 and three new edges x_1c_1, x_2c_2, x_3c_3, and so x_1, x_2, x_3 are the leaves of Z (see Fig. 10).

Now we are ready to define the class of graphs $\mathcal{F}$ recursively. First we assume that Y and Z are in $\mathcal{F}$. Suppose that A and B are disjoint graphs such that A has a triangle T, $N(T,A) = \{a_1,a_2,a_3\}$, and $L(B) = \{b_1,b_2,b_3\}$. Let $A(T,B)$ be a graph obtained from A by *replacing* its triangle T by $B - L(B)$, i.e. $A(T,B)$ is obtained from $A - T$ and B by identifying each a_i with b_i, $i \in \{1,2,3\}$. Now if $A, B \in \mathcal{F}$, then we assume that also $A(T,B) \in \mathcal{F}$.

It is easy to prove that the graphs in $\mathcal{F}$ have the following simple properties.

4.3 *Let $G \in \mathcal{F}$ and $G \neq Y$. Then*
$(a1)$ $|L(G)| = 3$ *and if $x \in V(G - L(G))$, then $d(x,G) = 3$,*
$(a2)$ *G has triangles and if T is a triangle of G, then*
$(a2.1)$ $|N(T,G)| = |D(T,G) = 3$,
$(a2.2)$ *$N(T,G)$ induces in G the subgraph with no edges and $D(T,G)$ is a 3-edge*

cut-matching in G,

$(a2.3)$ *there is a unique 6-cycle* C *in* G *such that* $N(T,G) \subset V(C)$ *and* $D(T \cup C, G)$ *is a 3-edge cut-matching in* G (we put $C = C(T,G)$ and $D(T \cup C, G) = M(T,G)$).

If $F \in \mathcal{F}$, then let $\dot{F}$ denote the graph obtained from F by identifying the three leaves with a new vertex x, $\bar{F}$ the graph obtained from F by adding the triangle T with the vertex set $L(F)$ and $\ddot{F}$ the graph obtained from $\bar{F}$ by adding a new vertex z, by subdividing every edge e in T with a new vertex v_e, and by adding three new edges zv_e, $e \in E(T)$.

It is easy to see the following.

4.4 *Let* $F \in \mathcal{F}$. *Then* $\dot{F}$, $\bar{F}$, *and* $\ddot{F}$ *are cubic 3-connected graphs.*

Now we can describe some Λ-packing properties of F, $\dot{F}$, $\bar{F}$, and $\ddot{F}$ for $F \in \mathcal{F}$.

4.5 *Let* $F \in \mathcal{F}$. *Then*

$(a1)$ $v(F) = 0 \bmod 6$ *and* F *has no* Λ*-factor,*

$(a2)$ $v(\dot{F}) = 4 \bmod 6$ *and* $\dot{F} - (N(x,\dot{F}) \cup x \cup X)$ *has no* Λ*-factor for every* $X \subset V(G)$ *such that* $|X| = 3$ *and* X *is matched with* $N(x)$ *in* $\dot{F}$ (see also **2.4** $(a2)$ and Fig. 4),

$(a3)$ $v(\bar{F}) = 0 \bmod 6$ *and* $\bar{F} - E(T)$ *has no* Λ*-factor, where* T *is the triangle in* $\bar{F}$ *with* $V(T) = L(F)$, *and*

$(a4)$ $v(\ddot{F}) = 4 \bmod 6$ *and* $\ddot{F} - (N(z,\ddot{F}) \cup z)$ *has no* Λ*-factor.*

Proof (uses **2.4** $(a2)$ and **4.3**). Claims $(a2)$, $(a3)$, and $(a4)$ follow from $(a1)$. We prove calim $(a1)$. Obviously, $v(F) = 0 \bmod 6$ and our claim is obviously true for Y and Z. Suppose, on the contrary, that $(a1)$ is not true. Let G be a vertex minimum counterexample, and so $G \in \mathcal{F}$ and G has a Λ-factor, say P. By definition of $\mathcal{F}$, we have: $G = A(T,B)$ for some $A, B \in \mathcal{F}$ and a triangle T in A. By **4.3**, there exist $M = M(T,A)$ and $C = C(T,A)$. Obviously, $v(B \cup S)) = 0 \bmod 3$. Therefore (P,M) satisfies one of the conditions in **2.4** $(a2)$ (see Fig. 4). Let $Q = P \cup S$ and $B' = (B \cup C) - Q$. Then $P_1 = P \cap B'$ is a Λ-factor in B' and $P_2 = P - P_1$ is a Λ-factor in $G - B'$.

Suppose that (P,M) satisfies conditions $(a2.2)$ with $E(S) \cap E(P) \ne \emptyset$ (and so $|E(C) \cap E(P)| \in \{1,3\}$) or $(a2.1)$ or $(a2.2)$. Then $T' = (T \cup C \cup D(T,A)) - Q$ has a Λ-factor P'_1. Therefore $P'_1 \cup P_2$ is a Λ-factor in A. However, $A \in \mathcal{F}$ and $v(A) < v(G)$. Therefore the counterexample G is not vertex minimum, a contradiction.

Now suppose that (P,M) satisfies condition $(a2.2)$ with $E(C) \cap E(P) = \emptyset$. Then $B' = B$, and so P_1 is a Λ-factor in B. However, $B \in \mathcal{F}$ and $v(B) < v(G)$. Therefore again the counterexample G is not vertex minimum, a contradiction. $\square$

Now we describe a sequence (mentioned in the above remark $(r2)$) of cyclically 6-connected graphs G with two disjoint 3-vertex paths L, L' such that $v(G) = 0 \bmod 6$ and $G - (L \cup L')$ has no Λ-factor. Let C_s be a cycle with $9s$ vertices, $s \ge 1$ and let

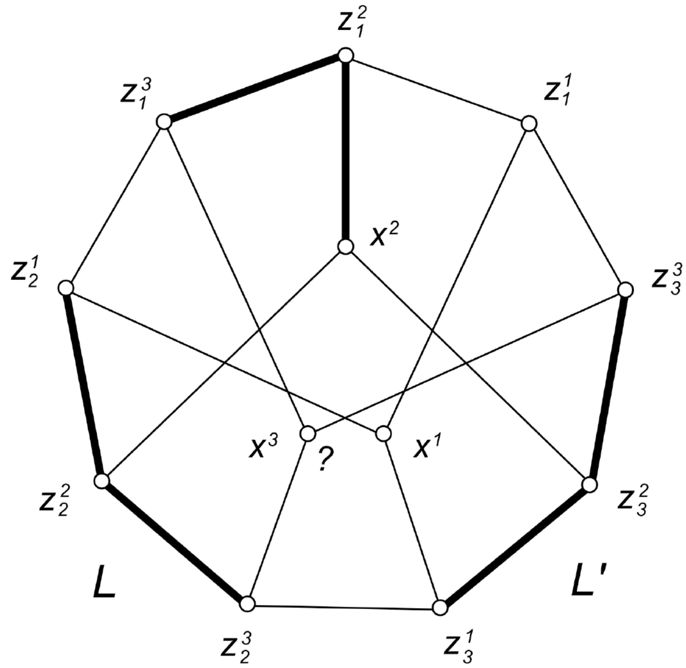

Figure 11: (R_1, L, L'), $v(R_1) = 12$

$\{L_k : k \in \{1, \dots, 3s\}$ be a Λ-factor of C_s, where $L_i = (z_i^1 z_i^2 z_i^3)$. Let R_s be the graph obtained from C_s by adding the set $\{x_i^j : i \in \{1, \dots, s\}, j \in \{1, 2, 3\}\}$ of $3s$ new vertices and the set $\{x_i^j z_i^j, x_i^j z_{i+s}^j, x_i^j z_{i+2s}^j : i \in \{1, \dots, s\}, j \in \{1, 2, 3\}\}$ of $9s$ new edges (see, for example, (R_1, L, L') in Fig. 11).

It is easy to prove the following

4.6 *Let R_s be the graph described above, $s \ge 1$, and $\{L, L'\} \subset \{L_i, L_{i+s}, L_{i+2s}\}$ for some $i \in \{1, \dots, s\}$. Then*

(a1) R_1 is a cubic cyclically 5-connected graph, R_s is a cubic cyclically 6-connected graph for $s \ge 2$, $v(R_s) = 12s$, and

(a2) $R_s - (L \cup L')$ has no Λ-factor.

Using operation $Aa\sigma bB$ (see Fig. 1), **2.4**, and **4.5**, it is easy to prove the following.

4.7 *There are infinitely many pairs (G, K) such that G is a cubic 3-connected graph, K is a 3-edge cut of G, $v(G) = 0 \bmod 6$, and $|E(P) \cap K| \notin \{0, 1\}$ for every Λ-factor P of G.*

5 On a Λ-factor homomorphism in cubic graphs

Let, as in Section 2, $G = B\{(A(u), a^u) : u \in V(B)\}$, $N(a^u, A(u)) = N^u = \{a_1^u, a_2^u, a_3^u\}$, $A^u = A(u) - a^u$, and $E' = E(G) \setminus \cup\{E(A^v) : v \in V(B)\}$. As we men-

tioned above, there is a unique bijection $\alpha : E(B) \to E'$ such that if $uv \in E(B)$, then $\alpha(uv)$ is an edge in G having one end-vertex in A^u and the other end-vertex in A^v.

Let $D(A^u, G) = D^u = \{a_i^u b_i^u : i \in \{1, 2, 3\}\}$, and so D^u forms a matching in G. Then $A^u \cup D^u$ is a subgraph of G. Let $A'(u)$ be the graph obtained from $A^u \cup D^u$ by adding the triangle T^u with $V(T^u = \{b_1^u, b_2^u, b_3^u\}$. Suppose that P is a Λ-factor of B and vuw is a 3-vertex path in P. We need the following additional notation:

$V_s(P)$ is the set of vertices of degree s in P (and so $s \in \{1, 2\}$),

$A_1(v, P) = A^v - End(\alpha(uv))$ and $A_2(u, P) = A'(u) - b$, where b is the vertex in $V(T^u)$ that is incident to no edge in $\{\alpha(us), \alpha(uv)\}$,

$\Gamma(G, P)$ is the set of Λ-factors Q of G such that $E(P) = \{\alpha^{-1}(e) : e \in E'(G) \cap E(Q)\}$,

$\Gamma(H)$ is the set of Λ-factors of a graph H, and

$X \bigotimes Y$ is the Cartesian product of sets X and Y.

It is not difficult to prove the following homomorphism theorem for Λ-factors in 3-connected graphs.

5.1 *Let B and each $A(u)$, $u \in V(B)$, be cubic 3-connected graphs. Suppose that each $v(A(u)) = 2 \bmod 6$ and each $(A(u), a^u)$ satisfies the following conditions:*

$(h1)$ $A(u) - (N^u \cup a^u \cup y)$ has no Λ-factor for every vertex y in $A(u) - (N^u \cup a^u)$ adjacent to a vertex in N^u and

$(h2)$ $A(u) - \{a^u, z\}$ and $A(u) - W$ has a Λ-factor for every $a^u z \in E(A(u))$ and a 5-vertex path W in $A(u)$ centered at a^u, respectively.

Then

$(\gamma 1)$ $\Gamma(G, P) \cap \Gamma(G, Q) = \emptyset$ for $P, Q \in \Gamma(B)$, $P \neq Q$,

$(\gamma 2)$ $\Gamma(G, P) = (\bigotimes\{\Gamma(A_1(v, P)) : v \in V_1(P)\})(\bigotimes\{\Gamma(A_2(u, P)) : u \in V_2(P)\})$, and

$(\gamma 3)$ $\Gamma(G) = \bigcup\{\Gamma(G, P) : P \in \Gamma(B)\}$,

and so G has a Λ-factor if and only if B has a Λ-factor.

By **4.1**, there are infinitely many pairs (A, a) such that A is cubic 3-connected graph, $v(A) = 2 \bmod 6$, $a \in V(A)$, and (A, a) satisfies $(h1)$ in **5.1**. If $(z1)$ is true, then by **3.1**, condition $(h2)$ in **5.1** is satisfied for every pair (A, a) such that A is cubic 3-connected graph, $v(A) = 2 \bmod 6$, and $a \in V(A)$.

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